First-order Linear Eq

$$\frac{dy}{dt} + p(t)\,y = g(t)$$

Such an eq. can be solved using integrating factors. The idea is to mult both sides by a certain fnc $\mu(t)$, chosen such the LHS looks like a product rule.

Multiply both sides by $\mu(t)$:

$$\boxed{\mu(t)\,y' + \mu(t)\,p(t)\,y = \mu(t)\,g(t)}$$

Choose $\mu(t)$ s.t. $\mu'(t) = \mu(t)\,p(t)$.

$\Rightarrow \dfrac{\mu'(t)}{\mu(t)} = p(t) \overset{\text{integrate}}{\Rightarrow} \ln \mu(t) = \int p(t)\,dt + C$

$\Rightarrow \boxed{\mu(t) = e^{\int p(t)\,dt + C} = e^{\int p(t)\,dt}}$ Take $C=0$

$$\frac{d}{dt}(\mu \cdot y) = \mu y' + \mu' y = \mu \cdot g$$

integrate $\Rightarrow \mu(t)\,y = \int \mu(t)\,g(t)\,dt + C$

$\Rightarrow \boxed{y = \dfrac{\int \mu(t)\,g(t)\,dt + C}{\mu(t)}}$ Gen. Sol.

1. Find the general solution to the following differential equations using the method of integrating factors:

   (a) $ty' - y = t^2 e^t$

   (b) $(1+t^2)y' + 4ty = (1+t^2)^{-2}$.

(a) $ty' - y = t^2 e^t$ → Not in general form

$\Rightarrow y' \boxed{-\frac{1}{t}}\, y = te^t$ — $p(t)$

Find $\mu(t)$: $\mu(t) = e^{\int -\frac{1}{t}dt} = e^{-\ln t} = e^{\ln t^{-1}} = \frac{1}{t}$

Multiply by $\mu(t)$.

$$\frac{1}{t}y' - \frac{1}{t^2}y = e^t$$

Integrate:

$$\frac{1}{t}y = \int e^t\,dt + C = e^t + C$$

Solve for y:

$\boxed{y = te^t + Ct}$ Gen Sol'n

(b) $(1+t^2)y' + 4ty = (1+t^2)^{-2}$

Not in general form

$\rightarrow y' + \boxed{\frac{4t}{1+t^2}}\, y = (1+t^2)^{-3}$ → $p(t)$

$\Rightarrow \mu(t) = e^{4\int \frac{t}{1+t^2}dt} = e^{2\ln(1+t^2)} = (1+t^2)^2$

$\int \frac{t}{1+t^2}dt$ $\quad u = 1+t^2,\ du = 2t\,dt$

$= \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln u = \frac{1}{2}\ln(1+t^2)$

$\Rightarrow (1+t^2)^2 y' + 4t(1+t^2)y = (1+t^2)^{-1}$

$\Rightarrow (1+t^2)^2 y = \int (1+t^2)^{-1} + C = \tan^{-1} t + C$

$\Rightarrow \boxed{y = \dfrac{\tan^{-1} t + C}{(1+t^2)^2}}$ Gen. Sol'n

2. Consider the following initial value problem: $y' - \frac{1}{2}y = 2\cos t$ with $y(0) = a$.

Skip (a), (b)

   (a) Use this GeoGebra tool to plot a direction field and several solution curves. (Don't try to do this part by hand!)

   (b) Notice that the end behavior (as $t \to \infty$) depend on a. Try to estimate the value of a for which the transition in end behavior occurs.

   (c) Solve the initial value problem using integrating factors. Find the exact value of a from part(b).

(c) $y' - \frac{1}{2}y = 2\cos t, \quad y(0) = a$

$\mu(t) = e^{-\int \frac{1}{2}dt} = e^{-\frac{1}{2}t}$

$\rightarrow e^{-\frac{1}{2}t}y' - \frac{1}{2}e^{-\frac{1}{2}t}y = -e^{-\frac{1}{2}t}\cos t$

$\Rightarrow \boxed{e^{-\frac{1}{2}t}y = -\int e^{-\frac{1}{2}t}\cos t\,dt + C}$

$\int u\,dv = uv - \int v\,du$

$u = e^{-\frac{1}{2}t}$ $\quad v = \sin t$

$du = -\frac{1}{2}e^{-\frac{1}{2}t}$ $\quad dv = \cos t\,dt$

$u = e^{-\frac{1}{2}t}$ $\quad v = -\cos t$

$du = -\frac{1}{2}e^{-\frac{1}{2}t}$ $\quad dv = \sin t\,dt$

$\int e^{-\frac{1}{2}t}\cos t\,dt = e^{-\frac{1}{2}t}\sin t + \frac{1}{2}\int e^{-\frac{1}{2}t}\sin t\,dt$

$= e^{-\frac{1}{2}t}\sin t + \frac{1}{2}\left[-e^{-\frac{1}{2}t}\cos t - \frac{1}{2}\int e^{-\frac{1}{2}t}\cos t\right]$

$= e^{-\frac{1}{2}t}\left(\sin t - \frac{1}{2}\cos t\right) - \frac{1}{4}\int e^{-\frac{1}{2}t}\cos t\,dt$

$\rightarrow \frac{5}{4}\int e^{-\frac{1}{2}t}\cos t\,dt = e^{-\frac{1}{2}t}\left(\sin t - \frac{1}{2}\cos t\right)$

$\Rightarrow \boxed{\int e^{-\frac{1}{2}t}\cos t\,dt = \frac{4}{5}e^{-\frac{1}{2}t}\left(\sin t - \frac{1}{2}\cos t\right) + C}$

$\Rightarrow \boxed{y = \frac{4}{5}\left(\sin t - \frac{1}{2}\cos t\right) + Ce^{\frac{1}{2}t}}$

Gen Sol'n

Using $y(0) = a$ →

$$a = \frac{4}{8}\left(0 - \frac{1}{2}\right) + C$$

$\Rightarrow \boxed{C = a + \frac{2}{8}}$